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CONSTRUCTIVE CHARACTERIZATIONS OF THE VALUE FUNCTION OF A MIXED-INTEGER PROGRAM II

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1. Introduction

We study the 'pre-multiplied mixed integer constraint set', which has the form:

$$(\mathrm{PMIP})_v \qquad Ax+By=Cv, \qquad x,y\geq 0; \quad x \text{ integer}.$$

We shall provide a characterization of the sets of the form

$$S=\{v \mid \text{for some } x,y\geq 0 \text{ with } x \text{ integer, } Cv=Ax+By\}$$

in terms of an inductively-defined class of functions, the Gomory functions of [2] (see Theorems 3.1 and 4.4 below). We shall also obtain characterizations of 'value functions' of the form

$$z(v)=\min\{cx+dy \mid Ax+By=Cv;\ x,y\geq 0;\ x \text{ integer}\}$$

in terms, both, of an inductively-defined function class, and a second function class described below (see Theorem 4.4 and Corollary 4.7 below). In our work, we shall assume that A, B, C, v, c and d are rational.[1]

Constraint sets of the form $(\mathrm{PMIP})_v$ occur frequently in practice. If $C=I$ (the identity matrix), or more generally if C is invertible, then $(\mathrm{PIMP})_v$ is the usual mixed-integer program. Also, in a mixed-integer program in which right-hand-side (r.h.s.) ranging analysis is desired, and material balance equations are present which need the r.h.s. kept at zero throughout the analysis, the constraint set has the form

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[1] Most of our results hold in the case when v, c, d are arbitrary reals. However, the rationality of A, B, C is crucial to our work.

$(\mathrm{PMIP})_v$ in which every row of C is either a unit vector or the zero vector (and such C are not invertible). More generally, if ranging analysis is needed for all r.h.s. of a mixed-integer program which also lie in a polyhedral cone given by generators, the constraint set $(\mathrm{PMIP})_v$ arises.

This paper continues our investigations in [4] regarding mixed integer programs. A crucial difference between $(\mathrm{PMIP})_v$ and the mixed-integer program is that $(\mathrm{PMIP})_v$ yields to characterizations by inductively-defined classes of functions, while – as we showed in [4] via several examples – the mixed-integer program does not. For instance, the sum of two value functions, or the maximum of two value functions, of a mixed-integer program may fail to be the value function of a mixed-integer program (see [4]). This major difficulty – i.e., the lack of an inductive structure – did not occur in earlier investigations of linear programs or pure integer programs (the case that B and d are empty), since one can show (although it is non-trivial to show this) that the 'premultiplied' linear or pure-integer program reduces to its ordinary form. However, for the mixed-integer program, the presence of discrete along with continuous variables forces us to make a distinction.

In outline, the present paper proceeds as follows. Section 2 summarizes the relevant background and provide definitions to be used later. In Section 3, we provide our characterization of sets of the form S as above, via functions which are 'consistency testers' (defined in Section 2). We also characterize (in a non-algorithmic way) which sets S arise from the case $C=I$ of a mixed-integer program $(\mathrm{MIP})_v$. In Section 4, we treat value functions $z(v)$, with a special result for the case $d=0$.

The results here appeared earlier in [3].

2. Definitions and background

Sets of the form $S = \{v \mid \text{for some } x, y \geq 0 \text{ with } x \text{ integer, } Cv = Ax + By\}$ are called *finitely generated mixed-integer monoids,* abbreviated f.g.m.m.

The *infimal convolution* [15] of a finite set $f_1, \dots, f_2$ of functions, denoted $\mathrm{infcon}\{f_1, \dots, f_t\}$, is the function f defined by

$$f(v) = \inf\{f_1(v^1) + \cdots + f_t(v^t) \mid v^1 + \cdots + v^t = v\}. \tag{2.1}$$

It is possible to have $f(v) = -\infty$ even if all f_i are finite-valued. One easily proves that only the infimal convolution of functions two-at-a-time need be defined.

To handle occurrences of $-\infty$ which can arise due to infimal convolution, we shall adhere to these conventions, whenever $r \in R \cup \{-\infty\}$:

$$\begin{aligned} &(-\infty)\cdot r = r\cdot(-\infty) = -\infty \quad \text{if } r>0;\\ &(-\infty)\cdot 0 = 0\cdot(-\infty) = 0;\\ &r + (-\infty) = (-\infty) + r = -\infty. \end{aligned} \tag{2.2}$$

The infimal convolution of functions f and g will be abbreviated $\mathrm{infcon}\{f, g\}$.

The *mixed Gomory functions* is the class of functions inductively defined by these clauses:

(2.3a) All linear functions λv with $\lambda \in Q^m$ (Q = the rational) are mixed Gomory.

(2.3b) If F_1 and F_2 are mixed Gomory and α and β are non-negative rationals, then $\alpha F_1 + \beta F_2$ is mixed Gomory.

(2.3c) If F is mixed Gomory, then $\lceil F \rceil$ is mixed Gomory.

(2.3d) If F_1 and F_2 are mixed Gomory, then $\max\{F_1, F_2\}$ is mixed Gomory.

(2.3e) If F_1 and F_2 are mixed Gomory, then $\operatorname{infcon}\{F_1, F_2\}$ is mixed Gomory.

In clause (2.3c), the *round-up* $\lceil F \rceil$ of F is the function given by $\lceil F \rceil(v)$ = the least integer which equals or exceeds $F(v)$.

The *Gomory* functions are defined analogously, using only the clauses (2.3a) to (2.3d), with 'mixed Gomory' changed to 'Gomory' in every occurence. The *Chvátal* functions are defined by clauses (2.3a)–(2.3c), with 'mixed Gomory' replaced by 'Chvátal' in every occurence. Thus all Chvátal functions are Gomory functions, and all Gomory functions are mixed-Gomory functions. (However, it can be shown that both reverse inclusions fail.)

Note that the mixed Gomory functions take values in $R \cup \{-\infty\}$. This class of functions can alternatively be defined, as inductively obtained by closing the class of Gomory functions under infimal convolution.

One other class of functions is relevant. The *monotone Gomory functions* are defined by (2.3a)–(2.3d), where 'mixed Gomory', is replaced by 'monotone Gomory', and '$\lambda \in Q^m$' in (2.3a) is replaced by '$\lambda \in Q^m$, $\lambda \geq 0$'. Clearly, all monotone Gomory functions are Gomory functions.

Let S be a set and G be a function. Then we shall say that G is a *consistency-tester* for S if

(2.4) $v \in S \Leftrightarrow G(v) \leq 0$

The terminology is motivated by the following consideration: if

$$S = \{v \mid \text{for some } x, y \geq 0 \text{ with } x \text{ integer, } Cv = Ax + By\},$$

then G is a consistency tester for S iff

(2.5) $(\text{PMIP})_v$ is consistent $\Leftrightarrow G(v) \leq 0$.

We may then say that "G is a consistency tester for $(\text{PMIP})_v$", and the phrase is used similarly for a mixed-integer program $(\text{MIP})_v$.

We shall need the following results.

Theorem 2.1 (see [4, Theorem 3.2]). *The consistency tester for* $(\text{MIP})_v$ *is a Gomory* *function. Moreover, if $d=0$ in the criterion function $cx+dy$ for*

$$z(v) = \inf\{cx+dy \mid Ax+By = v;\ x, y \geq 0;\ x, y \geq 0;\ x \text{ integer}\},$$

then the value function $z(v)$ is a Gomory function on its domain of definition

$$S = \{v \mid \text{for some } x, y \geq 0 \text{ with } x \text{ integer, } v = Ax + By\}.$$

Proposition 2.2 (see [4, Lemma 5.2]). *Let $H: R^k \Rightarrow R$ be a Gomory function. There are $\lambda^1, \ldots, \lambda^N \in Q^k$ and a monotone Gomory function $F: R^N \Rightarrow R$ such that:*
(i) $G(v)=F(\lceil\lambda^1 v\rceil, \ldots, \lceil\lambda^N v\rceil) \leq 0$ *iff* $H(v) \leq 0$.
(ii) $F(e_i)>0$, $1 \leq i \leq N$ *(where e_i is the ith unit vector).*

Theorem 2.3 (see [2, Theorems 5.1 and 5.2]). *For A, c rational put*

$$z(v) = \inf\{cx \mid Ax=v,\ x \geq 0,\ x \text{ integer}\},$$

and let $(\text{IP})_v$ *denote the constraint set of the program indicated.*
If $z(\mathbf{0})=0$, there are Gomory functions F, G such that:
(i) $(\text{IP})_v$ *is consistent iff $G(v) \leq 0$, i.e. G is a consistency tester for* $(\text{IP})_v$.
(ii) *If $G(b) \leq 0$, then $F(b)=z(b)$.*

Theorem 2.4 (see [2, Theorem 3.13]). *Let F, G be Gomory functions. Suppose that $G(b)>0$ whenever b is not integer.*
Then there are rational A, c such that (i) *and* (ii) *of Theorem* 2.3 *hold.*

We recall that a function F on a domain D is called *subadditive* if for all $v, w \in D$

(2.6) $$F(v+w) \leq F(v)+F(w).$$

By convention, the domain D of a subadditive function must be a monoid (i.e. $0 \in D$ and D is closed under addition). A monoid is called *finitely-generated* if it arises as non-negative integral combinations of a finite set of vectors (the latter are then called *generators*). If a monoid is finitely generated with integer vectors as generators, it is termed a finitely generated integer monoid [11].

Proposition 2.5 ([11]). *Gomory functions, value functions of integer programs, and value functions of mixed-integer programs are defined on monoids and are all subadditive on their domains.*

Theorem 2.6 (see [4, Theorem 6.1]). *The value function of a mixed integer program which is finite-valued for a r.h.s. of $b=0$, is the minimum of finitely many Gomory functions where it is defined.*

3. Consistency testers for $(\text{PMIP})_v$

Theorem 3.1. *The Gomory functions provide exactly a class of consistency testers* *for the finitely-generated mixed integer monoids.*

Moreover, the sets S for which there is a rational matrix Λ and a finitely generated integer monoid M with

$$S = \bigcup_{m \in M} \{v \mid \Lambda v \le m\} \tag{3.1}$$

are exactly the finitely-generated mixed monoids.

Proof. It suffices to show, in order, these implications: if S is a finitely generated mixed monoid, its consistency tester is a Gomory function; if G is a Gomory function, then there is a rational matrix Λ and a finitely generated integer monoid M such that (3.1) holds, for $S=\{v \mid G(v)\le 0\}$; if (3.1) holds, then S is a finitely generated mixed monoid.

By Theorem 2.1, there is a Gomory function F such that $F(Cv)\le 0$ iff $(\text{PMIP})_v$ is consistent. Then $G(v)=F(Cv)$ is a Gomory function. This establishes the first implication.

As to the second implication, let G be a Gomory function. Since the second implication concerns G only through the set $S=\{v \mid G(v)\le 0\}$, by Proposition 2.2, we may assume that G has the form

$$G(v) = F(\lceil \lambda^1 v \rceil, \ldots, \lceil \lambda^t v \rceil) \tag{3.2}$$

where $F(w)=F(w_1,\ldots,w_t)$ is a monotone Gomory function.

We define $H: R^t \Rightarrow R$ by

$$H(w) = \max\{F(w),\ \lceil w_i \rceil - w_i,\ i=1,\ldots,t\} \tag{3.3}$$

$H(w)\le 0$ iff w is an integer vector and $F(w)\le 0$. We apply Theorem 2.4 with $F=0$, $G=H$ to obtain an integer program such that $H(w)\le 0$ iff $(\text{IP})_v$ is consistent. We let M be the monoid generated by the columns of A and let Λ be the matrix with ith row λ_i, $1\le i\le t$ and (3.1) holds.

As to the third implication, suppose that (3.1) holds. Then if M is generated by m^j for $j=1,\ldots,s$ we have:

$$\begin{aligned} v\in S &\Leftrightarrow \text{for some } m\in M,\ \Lambda v\le m \\ &\Leftrightarrow \text{there exist } x,y\ge 0 \text{ with } x \text{ integer and } \Lambda v = \sum_{j=1}^{s} m^j x_j - Iy \end{aligned} \tag{3.4}$$

where I is the identity matrix. Thus, in $(\text{PMIP})_v$ we may take $C=\Lambda$, $B=-I$, and $A=[m^j]$ (columns). □

Theorem 3.2. *A set S is exactly the set of feasible r.h.s. v of a mixed integer program* $(\text{MIP})_v$, *if and only if there exists a finitely generated integer monoid M and a matrix of rationals Λ which, both satisfy* (3.1), *and are such that the equality system*

$$Av = m \tag{3.5}$$

has a solution for every $m \in M$.

Proof. Suppose that A, B are rational matrices with

(3.6) $\qquad v \in S \Leftrightarrow v = Ax + By$ for some $x, y \geq 0$ with x integer.

Let $\lambda^1, \ldots, \lambda^t$ be a finite set of generators for the cone $\{\lambda \mid \lambda B \leq 0\}$. Without loss of generality the λ^i are integer vectors such that $\lambda^i A$ is integer for $i = 1, \ldots, t$. Then $w = By$ for some $y \geq 0$ iff $\lambda^i w \leq 0$ for $i = 1, \ldots, t$. We have:

(3.7)
$$\begin{aligned} v \in S &\Leftrightarrow \text{for some } y \geq 0,\ v - Ax = By \text{ for some integer } x \geq 0 \\ &\Leftrightarrow \text{for some } x \geq 0 \text{ integer},\ \lambda^i Ax \geq \lambda^i v \text{ for } i = 1, \ldots, t \\ &\Leftrightarrow \text{for some } m \in M,\ m \geq \Lambda v \end{aligned}$$

where M is the integral monoid generated by the $\lambda^i A$ and where $\Lambda = [\lambda^i]$ (rows). Hence, (3.1) holds. Also, if $m = \Lambda A x \in M$ for $x \geq 0$ integer, then $v = Ax$ solves (3.5).

For the converse, suppose that (3.1) holds and that (3.5) is solvable for all $m \in M$. Observe that there is a rational matrix Γ with this ('pseudo-inverse') property, which can easily be constructed from Smith Normal Form:

(3.8) $\qquad \Lambda v = d$ is solvable $\Rightarrow \Lambda(\Gamma d) = d$.

Let $m^1, \ldots, m^u$ be a finite integer basis for the integer monoid M and let $v^1, \ldots, v^s$ be a finite rational basis for the cone $\{v \mid \Lambda v \leq 0\}$. Then

(3.9)
$$\begin{aligned} v \in S &\Leftrightarrow \text{for some } m \in M,\ \Lambda v \leq m \\ &\Leftrightarrow \text{for some } m \in M,\ \Lambda v \leq \Lambda \Gamma m \\ &\Leftrightarrow \text{for some } m \in M,\ \Lambda(v - \Gamma m) \leq 0 \\ &\Leftrightarrow \text{for some } y_1, \ldots, y_j \geq 0, \text{ and some } m \in M, \\ &\qquad v - \Gamma m = \sum_{j=1}^{s} y_j v^j \\ &\Leftrightarrow \text{for some integer } x_1, \ldots, x_u \geq 0 \text{ and scalars } y_1, \ldots, y_j \geq 0 \\ &\qquad v = \sum_{k=1}^{u} (\Gamma m^k) x_k + \sum_{j=1}^{s} v^j y_j. \end{aligned}$$

We have thus constructed a mixed integer program for which S is the set of feasible r.h.s. $\square$

We now provide a perspective for the results of this section.

From our earlier work, as cited above in Theorems 2.3 and 2.4, we saw that the Gomory functions provide consistency testers for pure integer programs, but that a special hypothesis (i.e. $G(b) > 0$ for b not integer) was needed to know that any specific Gomory function was a consistency tester for a pure IP. Theorem 3.1 of this section elucidates the need for the special hypothesis: indeed, 'most' Gomory functions are consistency testers for a PMIP, but not a pure IP.

In Part I of this paper [4], we provided a finite algorithm for determining when a given Gomory function G is the consistency tester for an MIP (as opposed to a

PMIP). The procedure given was of high complexity (it is neither NP nor co-NP), but one part of it was relatively easy, i.e. it was not hard to identify the continuous columns of the MIP. Indeed, if G were a consistency tester for an MIP, then v is of the form By for some $y \geq 0$ iff $G(\delta v) \leq 0$ for all $\delta \geq 0$. By (the inductive construction in [2, Lemma 5.2]) for Proposition 2.2, which allows us to assume that G has the form $G(v) = F(\lceil \lambda^1 v \rceil, \ldots, \lceil \lambda^t v \rceil)$, the latter condition is equivalent to: $\lambda^i v \leq 0$ for $i = 1, \ldots, t$. In terms of the proof of Theorem 3.2 of this section, $\lambda^1, \ldots, \lambda^t$ are then one basis for the cone $C = \{\lambda \mid \lambda B \leq 0\}$ and $\Lambda = [D\lambda^i]$ (rows) may be used as the matrix occuring in (3.5), for some suitable positive integer $D \geq 1$ (i.e. any D with $D\lambda^i A$ integer for $i = 1, \ldots, t$).

By Theorem 3.2, if there is then a monoid M such that (3.5) has a solution for every $m \in M$, G is indeed a consistency tester for some MIP. Thus Theorem 3.2 provides an illuminating (but not algorithmic) sufficient condition to supplement the necessary and sufficient conditions associated with our procedure from [4].

4. Value functions for $(\text{PMIP})_v$

We next turn our attention to the value function of $(\text{PMIP})_v$ with respect to a chosen criterion function $cx + dy$, i.e., functions of the form

$$z(v) = \inf\{cx + dy \mid Ax + By = Cv \text{ for some } x, y \geq 0 \text{ with } x \text{ integer}\},$$

where A, B and C are rational matrices.

From Example 7.1 in [4], we know that an MIP with only a single row may have value functions which are not Gomory functions. For that reason, certainly one must enlarge the class of Gomory functions in order to account for value functions of PMIP. In this section, we shall see that the class of mixed Gomory functions is the proper enlargement. We will show that, in terms of distance measures, mixed Gomory functions are 'very close' to Gomory functions (see Corollary 4.11 below). Moreover, our proofs below reveal that the infimal convolution operation need only be used once.

Let two finitely generated mixed monoids be defined by:

(4.1a) $\quad S_1 = \{v \mid A_1 x + B_1 y = C_1 v;\ x, y \geq 0;\ x \text{ integer}\}$

(4.1b) $\quad S_2 = \{v \mid A_2 x + B_2 y = C_2 v;\ x, y \geq 0;\ x \text{ integer}\}$

for rational matrices A_i, B_i, C_i and $i = 1, 2$.

Lemma 4.1. *Suppose that $z_1(b)$ and $z_2(b)$ are pre-multiplied value functions (where one or both functions may be identically $-\infty$ where defined).*

Then the following functions are pre-multiplied value functions (including possibly the $-\infty$ value function):

(a) $\alpha z_1 + \beta z_2$ *when $\alpha, \beta \geq 0$ are rational scalars,*

(b) $\max\{z_1, z_2\}$,

(c) $\lceil z_1 \rceil$,
(d) $\operatorname{infcon}\{z_1, z_2\}$.

Proof. For notational purposes, set:

(4.2) $$z_i(b) = \inf\{c^i x + d^i y \mid A^i x + B^i y = C^i b;\ x, y \geq 0;\ x \text{ integer}\}, \quad i = 1, 2,$$

where A^i, B^i, C^i are rational matrices and c^i and d^i are rational vectors.

All the above results (a) to (d) are proven by first writing these constraints:

(4.3) $$\begin{aligned} &A^1 x + B^1 y - C^1 \beta^1 + C^1 \gamma^1 = 0; \\ &A^2 u + B^2 v - C^2 \beta^2 + C^2 \gamma^2 = 0; \\ &x, u, y, v, \beta^1, \beta^2, \gamma^1, \gamma^2 \geq 0; \quad x, u \text{ integer} \end{aligned}$$

and then appending some further constraints and an objective function z.

For (a), we append the constraints

$$\begin{aligned} &z - \alpha c^1 x - \alpha d^1 y - \beta c^2 u - \beta d^2 v = 0; \\ &\beta^1 - \gamma^1 = b, \quad \text{and} \quad \beta^2 - \gamma^2 = b. \end{aligned}$$

For (b), we append the two constraints:

$$\begin{aligned} &z - c^1 x - d^1 y \geq 0, \qquad z - c^2 x - d^2 y \geq 0, \\ &\beta^1 - \gamma^1 = b, \quad \text{and} \quad \beta^2 - \gamma^2 = b. \end{aligned}$$

For (c), we append the constraints:

$$\beta^2 = \gamma^2 = 0, \qquad \beta^1 - \alpha^1 = b \quad \text{and} \quad z \geq c^1 x + d^1 y;$$

and we also require that z be an integer variable.

For (d), we append the constraints

$$z - c^1 x - d^1 y - c^2 u - d^2 v = 0 \quad \text{and} \quad \beta^1 - \gamma^1 + \beta^2 - \gamma^2 = b.$$

Since all right-hand-sides are, in all cases (a) to (d), either zero of a component of b, we do have a pre-multiplied constraint set.

We leave it to the reader to check that the constraint sets in (a) to (d) define the value functions desired. □

Lemma 4.2. *Suppose that the function*

(4.4) $$g(v) = \min_{i=1,\ldots,t} f_i(v)$$

is subadditive. Then

(4.5) $$g = \operatorname{infcon}\{f_1, \ldots, f_t\}.$$

Proof. Let $i = 1, \ldots, t$. Using (2.1) with $v^i = v$ and $v^j = 0$ for $j \neq i$, we find that $f(v) \leq f_i(v)$. Since $i = 1, \ldots, t$ is arbitrary, we have $f(v) \leq g(v)$ for all v, where $f = \operatorname{infcon}\{f_1, \ldots, f_t\}$.

Now suppose that g is subadditive. Let $v^1, \dots, v^t$ be chosen arbitrarily so that $v^1 + \cdots + v^t = v$. We have

$$g(v) \le g(v^1) + \cdots + g(v^t) \le f_1(v^1) + \cdots + f_t(v^t). \tag{4.6}$$

Taking the infimum on the right in (4.6), we obtain $g(v) \le f(v)$ where $f = \operatorname{infcon}\{f_1, \dots, f_t\}$, using (2.1). Hence $g(v) = f(v)$ for all v. □

Lemma 4.3. *The value $z(v)$ of a pre-multiplied constraint set* (PMIP)$_v$ *is equal to the minimum of finitely many Gomory functions where $z(v)$ is defined, provided that $z(\mathbf{0}) > -\infty$. Furthermore, this minimum of Gomory functions will be subadditive on R^m.*

Proof. If $z(v) = \min\{cx + dy \mid Ax + By = Cv;\ x, y \ge 0;\ x \text{ integer}\}$, by Theorem 2.6 and [5, I, Theorem 4.6] there are finitely many Gomory functions $F_1, \dots, F_t$ with $G(v) \overset{\text{def}}{=} \min\{F_1(Cv), \dots, F_t(Cv)\}$ and each function $F_i(Cv)$ is a Gomory function.

By Proposition 2.5, G is subadditive. Hence for v^1 and v^2 given,

$$\begin{aligned} z(v^1 + v^2) &= G(C(v^1 + v^2)) = G(Cv^1 + Cv^2) \\ &\le G(Cv^1) + G(Cv^2) = z(v^1) + z(v^2) \end{aligned}$$

when both $z(v^i)$ are defined. Thus, $z(v)$ is subadditive. □

The next result is our main characterization theorem for constraints of the form (PMIP)$_v$ with criterion functions.

Theorem 4.4. *If $z(v) = \min\{cx + dy \mid Ax + By = Cv;\ x, y \ge 0;\ x \text{ integer}\}$ is the value function of a rational pre-multiplied constraint set, then there is a mixed Gomory function F and a Gomory function G such that:*

(i) (PMIP)$_v$ *is consistent iff $G(v) \le 0$.*

(ii) *When $G(v) \le 0$, $z(v) = F(v)$.*

Conversely, given an arbitrary mixed Gomory function F and an arbitrary Gomory, or mixed-Gomory, function G, there are rational A, B, C, c and d such that (i) *and* (ii) *hold.*

Proof. In the proof of the forward (i.e. first paragraph) implication, the existence of G is assured by Theorem 3.1.

Let $z(v)$ be a value function of a pre-multiplied constraint set. If $z(0) = -\infty$, then there are $x, y \ge 0$ with x integer, satisfying $cx + dy < 0$, $Ax + By = 0$. Hence $z(v)$ equals the mixed Gomory function

$$\operatorname{infcon}\{F_1(v^1) + F_2(v^2) \mid v^1 + v^2 = v\}$$

where $F_1(v^1) = \sum_j v_j^1$ and $F_2(v^2) = -2\sum_j v_j^2$ (as the latter infimal convolution is identically $-\infty$).

If $z(\mathbf{0}) > -\infty$, the desired form for $z(v)$ follows from Lemmas 4.2 and 4.3.

For the converse, let F and G be arbitrary mixed Gomory functions. If $F(v)=\lambda v$, F is the value function of the linear program

$$(4.7)\qquad \begin{aligned} &\inf\ \lambda(y^1-y^2),\\ &\text{subject to } y^1-y^2=v,\\ &\qquad y^1,y^2\geq 0.\end{aligned}$$

For the cases (2.3b) through (2.3e) in the definition of a mixed Gomory function, Lemma 4.1 applies, and we see inductively that F is the value function of some pre-multiplied constraint set; say

$$F(v)=\inf\{c^1x+d^1y \mid A^1x+B^1y=C^1v;\ x,y\geq 0;\ x \text{ integer}\}.$$

Since $F(v)<+\infty$ for all v, the constraint set exhibited is consistent for all v.

By the same reasoning, there are rational A^2, B^2, C^2, c^2, d^2 such that

$$G(v)=\inf\{c^2x+d^2y \mid A^2x+B^2y=C^2v;\ x,y\geq 0;\ x \text{ integer}\}.$$

Hence we have (using [12]):

$$(4.8)\qquad G(v)\leq 0 \Leftrightarrow \text{there are } u,\ w\geq 0 \text{ with } u \text{ integer},\quad c^2u+d^2w\leq 0,\quad A^2u+B^2w=C^2v.$$

The following is a pre-multiplied constraint set:

$$(4.9)\qquad \begin{aligned} &A^1x+B^1y=C^1v;\\ &A^2u+B^2w=C^2v;\\ &c^2u+d^2w\leq 0;\\ &x,y,u,w\geq 0;\quad x \text{ and } u \text{ integer}.\end{aligned}$$

One easily verifies that G is a consistency tester for (4.9), and that with criterion function c^1x+d^1y, F is the value function for (4.9). □

We list three consequences of Theorem 4.4 and its proof above, with remarks as necessary.

Corollary 4.5. *The mixed-Gomory functions are exactly the consistency testers for the pre-multiplied constraint sets.*

Remark. By Theorem 3.1 and this Corollary, two different classes of functions can be the consistency testers for the same class of constraints.

Corollary 4.6. *A mixed-Gomory function is either identically $-\infty$ or everywhere finite-valued.*

Remark. From the proof of Theorem 4.4, the property cited is a property of the value functions of premultiplied constraint sets that are always consistent.

Corollary 4.7. *The class of mixed Gomory functions F with $F(0) > -\infty$ is identical with both of the following two classes of functions:*

(a) *The infimal convolution of Gomory functions, such that this infimal convolution is finite-valued at the origin.*

(b) *The finite minimum of Gomory functions, such that the minimum is a subadditive function.*

Theorem 4.8. *Given mixed Gomory functions F and G, there exists a mixed integer program for which G is consistency tester and F is value function (i.e. Theorem* 4.4(i) *and* (ii) *hold with $C=I$) if and only if the mixed-Gomory function $H(z,v)=\max\{H'(z,v), G(v)\}$ is a consistency tester for a mixed integer program.*

In the above, $H'(z,v)$ is any mixed-Gomory function such that

$$H'(z,v) \le 0 \Leftrightarrow z \ge F(v). \tag{4.8}$$

Proof. Using the construction from Lemma 4.1 in the manner that it was cited in the proof of Theorem 4.4, we obtain rational A, B, C, c, d with

$$F(v) = \min\{cx+dy \mid Ax+By = Cv;\ x, y \ge 0;\ x \text{ integer}\}.$$

Then by [12]:

$$z \ge F(v) \Leftrightarrow \text{there are } x, y \ge 0 \text{ with } x \text{ integer,} \quad Ax+By = Cv, \quad cx+dy \le z. \tag{4.9}$$

By Theorem 3.1, H' exists for (4.8), since the right-hand part of (4.9) describes a pre-multiplied constraint set.

First, suppose there exists a mixed-integer program for which G is consistency tester and F is value function, and let it have criterion function $cx+dy$ and constraints $Ax+By=b$; $x, y \ge 0$; x integer. Then:

$$\begin{aligned} H(z,v) \le 0 &\Leftrightarrow z \ge F(v) \text{ and } G(v) \le 0 \\ &\Leftrightarrow \text{there are } x, y \ge 0 \text{ with } x \text{ integer and} \\ &\qquad cx+dy \le z, \quad Ax+By = v. \end{aligned} \tag{4.10}$$

Since the matrix of pre-multiplication on the right side of (4.10) is an identity matrix, H is the consistency tester of a mixed integer program.

Conversely, suppose H is the consistency tester for a mixed-integer program with constraints $Ax+By=b$, $cx+dy=z$; $x, y \ge 0$; x integer. We have:

$$\begin{aligned} G(b) \le 0 &\Leftrightarrow \text{for some } z, H'(z,b) \le 0 \\ &\Leftrightarrow H(z,b) \le 0 \text{ for some } z \\ &\Leftrightarrow \text{for some } z, \text{ there are } x, y \ge 0 \text{ with } x \text{ integer and} \\ &\qquad Ax+By = b,\ cx+dy = z \\ &\Leftrightarrow \text{there are } x, y \ge 0 \text{ with } x \text{ integer and} \\ &\qquad Ax+By = b. \end{aligned} \tag{4.11}$$

(For the direction $\Rightarrow$ of the first bi-conditional $\Leftrightarrow$ any $z \geq F(b)$ will do.) We also have, whenever $G(b) \leq 0$, that $H(z, b) \leq 0$ iff $z \geq F(b)$; hence

$$F(b) = \min\{cx + dy \mid Ax + by = b;\ x, y \geq 0;\ x \text{ integer}\}.$$

Therefore, G is consistency tester and F is value function for the constraints $Ax + By$. □

The next result identifies the instances of linear optimization over $(\mathrm{PMIP})_v$, in which the value function can be taken to be an integer-valued Gomory function (rather than the more complex mixed-Gomory function). A Gomory value function is equivalent to having $d = 0$ in the criterion function $cx + dy$; this is the case for which Gomory's mixed-integer algorithm [9] is guaranteed to be finite, assuming a lexicographically positive starting tableau.

Theorem 4.9. *If c is integer, $d = 0$ and $z(0) > -\infty$, then there is a Gomory function F such that*

$$F(v) = z(v) = \inf\{cx + dy \mid Ax + By = Cv;\ x, y \geq 0;\ x \text{ integer}\}$$

whenever $G(v) \leq 0$, where G is a Gomory function consistency tester for the constraints of the program indicated. Moreover, $F(v)$ is integer for all v.

Conversely given two Gomory functions F and G such that $F(v)$ is integer for all v, there is an optimization problem of the form indicated, with $d = 0$, c integer, and PMIP *constraints, for which G is consistency tester and F is value function. This converse also holds if F is a mixed Gomory function.*

Proof. First, assume that the program indicated is given. By Theorem 2.1, there is a Gomory function $H(b)$ such that $z(v) = H(Cv)$ whenever $G(v) \leq 0$, where G is the Gomory consistency tester guaranteed by Theorem 4.4. Then $F(v) = \lceil H(Cv) \rceil$ is a Gomory function, and $F(v) = z(v)$ whenever $G(v) \leq 0$. $z(v)$ is an integer since cx is integer for integer x.

Conversely, let a mixed Gomory function F and a Gomory function G be given, such that $F(v)$ is integer for all v, so $F(0) > -\infty$. By Theorem 4.4, there are rational A, B, C, c, d with

$$F(v) = \inf\{cx + dy \mid Ax + By = Cv;\ x, y \geq 0;\ x \text{ integer}\}$$

and G is a consistency tester for the constraints of the program indicated. Then:

$$\begin{aligned}(4.12)\qquad G(v) \leq 0 &\Leftrightarrow \text{there is a solution } x, y \geq 0,\ x \text{ integer, to} \\ &\qquad Ax + By = Cv \text{ with } cx + dy \text{ integer} \\ &\Leftrightarrow \text{there is a solution } x, y, z_1, z_2 \geq 0, \text{ with } x, z_1 \text{ and } z_2 \text{ integer,} \\ &\qquad \text{to } Ax + By = Cv,\quad z_1 - z_2 - cx - dy = 0.\end{aligned}$$

(The direction '$\Rightarrow$' of the first bi-condition '$\Leftrightarrow$' follows from the fact that $F(v)$ is integer for all v.) Thus, G is consistency tester for the premultiplied constraint set on the bottom right in (4.12) (where the premultiplication matrix consists of C augmented by a zero row). The value function of those constraints, for criterion for $z_1 - z_2$, is

$$\inf\{cx+dy \mid Ax+By=Cv;\ x,y\geq 0;\ x \text{ integer};\ cx+dy \text{ is integer}\} = F(v),$$

as $F(v)$ is integer. □

The requirement in Theorem 4.9, that $F(v)$ is integer for all v (or at least that $F(v)$ have the form n/D, where D is a fixed integer and n an integer depending on v), is essential to the converse part. Indeed, when $d=0$, $z(v)$ has the form n/D, for D a fixed integer. Thus, while the Gomory function $F(v)=v$ is trivially the value function of a PMIP (indeed, of an MIP), we must have $d\neq 0$ in all such PMIP.

Corollary 4.10. *If F is a mixed-Gomory function such that $F(v)$ is integer for all v, then F is a Gomory function.*

Proof. By Theorem 4.9. □

We conclude with a result that states that finite mixed Gomory functions are close, in the topology of uniform convergence, to Gomory functions.

Corollary 4.11. *For any $\varepsilon>0$ and any mixed Gomory function G with $G(0)>-\infty$, there is a Gomory function F which approximates G uniformly to within ε, from above:*

$$0 \leq F(v) - G(v) < \varepsilon \quad \textit{for all } v. \tag{4.13}$$

Proof. Let $D\geq 1$ be an integer such that $\varepsilon > 1/D$. We have that $H(v)=\lceil DG(v)\rceil$ is integer valued for all v. By Corollary 4.10, H is actually a Gomory function. Hence $F(v)=H(v)/D$ is a Gomory function, and we have 4.13. □

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